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# COMPONENTS OF MAXIMAL DIMENSION IN THE NOETHER-LEFSCHETZ LOCUS

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We will work over **C**. Let

$$Y = \{\text{algebraic surfaces of degree } d \text{ in } \mathbf{P}^3\},$$
$$\Sigma_d = \{S \in Y \mid S \text{ smooth and } \operatorname{Pic}(S) \text{ is not generated by the hyperplane bundle}\}.$$

We will call $\Sigma_d$ the *Noether-Lefschetz locus.* Some things that are known about $\Sigma_d$ are:

(1) $\Sigma_d$ has countably many irreducible components,

(2) For any irreducible component $\Sigma$ of $\Sigma_d$,

$$d-3 \leq \operatorname{Codim} \Sigma \leq \binom{d-1}{3}.$$

The upper bound on $\operatorname{codim} \Sigma_d$ is elementary, as this is just $h^{2,0}(S)$ (see [2]). The lower bound is more subtle and depends on fairly delicate algebraic considerations (see [4], [5]). One cannot do better for any $d \geq 3$, since the family $\Sigma_d^0$ of surfaces of degree $d$ containing a line has codimension exactly $d-3$ in $Y$. For $d=4$, the upper and lower bounds given in (2) coincide, so that every irreducible component of $\Sigma_d$ has codimension one. For higher $d$, the following result was conjectured in [2]:

**Theorem 1.** *For $d \geq 5$, the only irreducible component of $\Sigma_d$ having codimension $d-3$ is the family of surfaces of degree $d$ containing a line.*

It should be noted that Theorem 1 was obtained independently by Claire Voisin [7].

Let $\Sigma$ be an irreducible component of $\Sigma_d$ having codimension $d-3$. As shown in [5], if $S=\{F=0\}$ belongs to $\Sigma$, and $J_k(F)$ is the degree $k$ piece of the Jacobi ideal of $F$, generated by the first partials $F_0, F_1, F_2, F_3$ of $F$, then:

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*There exists a codimension* $(d-3)$ *linear subspace* $W \subseteq H^0(O_{\mathbf{P}^3}(d))$ *such that*

$$W \supseteq J_d(F),$$

and

(4) *The multiplication map* $W \otimes H^0(O_{\mathbf{P}^3}(d-4)) \to H^0(O_{\mathbf{P}^3}(2d-4))$ *is not surjective.*

(5) *The projection of* $W$ *into* $H^0(O_{\mathbf{P}^3}(d))/J_d(F)$ *is the Zariski tangent space to* $\Sigma$ *at* $S$.

We now introduce some notation. Given a linear subspace $W \subseteq H^0(O_{\mathbf{P}^r}(d))$ we let $\mu_k$ denote the multiplication map

$$W \otimes H^0(O_{\mathbf{P}^r}(k)) \xrightarrow{\mu_k} H^0(O_{\mathbf{P}^r}(d+k)),$$

and $c_k = \operatorname{codim}(\operatorname{im}\mu_k)$. We need the following algebraic result.

**Theorem 2.** *Let* $W \subseteq H^0(O_{\mathbf{P}^r}(d))$ *be a base-point free linear subspace of codimension* $c$. *If* $c \leq d$ *and* $c_{c-1} \neq 0$, *then*

(6) *for* $0 \leq k \leq c$, $\quad c_k = c - k$;

(7) *if* $r \geq 2$ *and* $d \geq c \geq 2$, *then* $W \supseteq I_d(L)$ *for some* $L \subseteq \mathbf{P}^r$.

*Proof of Theorem* 2. It was known to Macaulay (see [3], also [1], [6]) that for any $W \subseteq H^0(O_{\mathbf{P}^r}(d))$ of codimension $c$, if we write $c$ uniquely in the form

$$(8)\quad c = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_2}{2} + k_1, \quad (0 \leq k_1 < k_2 < \cdots < k_d),$$

where by convention $\binom{n}{m} = 0$ for $m > n$, then the image of the multiplication map $W \otimes H^0(O_{\mathbf{P}^r}(1)) \xrightarrow{\mu_1} H^0(O_{\mathbf{P}^r}(d+1))$ has $\operatorname{codim}(\operatorname{im}\mu_1) \leq c_{\langle d\rangle}$, where

$$(9)\qquad c_{\langle d\rangle} = \binom{k_d+1}{d+1} + \binom{k_{d-1}+1}{d} + \cdots + \binom{k_1+1}{2}.$$

Furthermore, it was shown by Gotzmann [3] that if equality holds, then

$$\operatorname{codim}(\operatorname{im}\mu_k) = (\cdots((c_{\langle d\rangle})_{\langle d+1\rangle})\cdots)_{\langle d+k-1\rangle}.$$

If $c \leq d$, then

$$k_d = d, k_{d-1} = d-1, \cdots, k_{d-c+1} = d-c+1,$$
$$k_{d-c} = d-c-1, \cdots, k_d = 1, k_1 = 0.$$

Thus

$$c_{\langle d\rangle} = \binom{d+1}{d+1} + \cdots + \binom{d-c+2}{d-c+2} = c.$$

By Gotzmann's result, if equality holds, then the image of

$$W \otimes H^0(O_{\mathbf{P}^r}(k)) \xrightarrow{\mu_k} H^0(O_{\mathbf{P}^r}(d+k))$$

always has codimension $c$. However, if $W$ is base-point free, then $\mu_k$ is surjective for $k$ sufficiently large. Proceeding inductively, if we let $c_k = \operatorname{codim}(\operatorname{im}\mu_k)$, then $c > c_1 > c_2 > \cdots$ for $W$ base-point free and $c \leq d$. Since by hypothesis $c_{c-1} \neq 0$, the only possibility is

$$c_k = c - k, \quad \text{for } 0 \leq k \leq c,$$

proving (6).

To prove (7), assume $d \geq c \geq 2$ and $r \geq 2$. We first notice that it is enough to prove that $W \supseteq I_d(H)$ for some hyperplane $H$. For if so, letting

$$W_H = \operatorname{im}(W \to H^0(O_H(d))),$$

$\mu_{k,H}$ be the multiplication map

$$W_H \otimes H^0(O_H(k)) \xrightarrow{\mu_{k,H}} H^0(O_H(d+k))$$

and

$$c_{k,H} = \operatorname{codim}(\operatorname{im}\mu_{k,H}),$$

we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & W \cap I_d(H) & \to & W & \to & W_H & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I_d(H) & \to & H^0(O_{\mathbf{P}^r}(d)) & \to & H^0(O_H(d)) & \to & 0 \end{array} \tag{10}$$

If $W \supseteq I_d(H)$, then $c_H = c$, and similarly $c_{k,H} = c_k$ for all $k \geq 0$. If $r = 2$, we are already done. If not, then by induction on $r$, $W_H$ contains the ideal of some line $L \subseteq H$. Then $W \supseteq I_d(L)$. Thus we are reduced to showing $W \supseteq I_d(H)$ for some hyperplane $H$.

Let $\mathbf{P}^{r^*}$ be the dual projective space and $J \subset \mathbf{P}^r \times \mathbf{P}^{r^*}$ the incidence correspondence

$$J = \{(P, H) \mid P \in H\}.$$

Let

$$\begin{array}{ccc} & J & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ \mathbf{P}^r & & \mathbf{P}^{r^*} \end{array}$$

be the projections. On $\mathbf{P}^r \times \mathbf{P}^{r^*}$, we have the exact sequence

$$0 \to f^*O_{\mathbf{P}^r}(-1) \otimes g^*O_{\mathbf{P}^{r^*}}(-1) \to O_{\mathbf{P}^r \times \mathbf{P}^{r^*}} \to O_J \to 0. \tag{11}$$

On $\mathbf{P}^r$, the evaluation map

$$W \otimes O_{\mathbf{P}^r} \to O_{\mathbf{P}^r}(d) \to 0$$

is surjective because $W$ is base-point free. Its kernel is therefore a vector bundle $M$ fitting into an exact sequence

$$0 \to M \to W \otimes O_{\mathbf{P}^r} \to O_{\mathbf{P}^r}(d) \to 0.$$

One readily sees that for $k \geq 0$,

$$H^0(O_{\mathbf{P}^r}(d+k))/\operatorname{im}\mu_k \cong H^1(M(k)).$$

Tensoring the sequence (11) with $f^*M(c-1)$, we obtain the exact sequence

$$0 \to f^*M(c-2) \otimes g^*O_{\mathbf{P}^{r*}}(-1) \to f^*M(c-1) \to O_J \otimes f^*M(c-1) \to 0.$$

Pushing down by $g$, we get a long exact sequence

$$\begin{aligned}\cdots \to H^1(M(c-2)) \otimes O_{\mathbf{P}^{r*}}(-1) \to H^1(M(c-1)) \otimes O_{\mathbf{P}^{r*}}\\ \to R^1_{g*}(O_J \otimes f^*M(c-1)) \to .\end{aligned}$$

If $h \in \mathbf{P}^{r^*}$ and $H \subseteq \mathbf{P}^r$ is the corresponding hyperplane, then

$$g^*O_h \otimes O_J \simeq O_H$$

and thus

$$H^q(g^*O_h \otimes O_J \otimes f^*M(c-1)) = 0 \quad \text{for } q \geq 2,$$

and

$$\begin{aligned}&H^1(g^*O_h \otimes O_J \otimes f^*M(c-1)) = 0\\ &\quad \leftrightarrow \text{ the multiplication map } W \otimes H^0(O_H(c-1))\\ &\qquad \to H^0(O_H(d+c-1)) \text{ is surjective.}\end{aligned}$$

Thus if $c_{c-1,H} = 0$ for every hyperplane $H$, then we obtain a surjective map of sheaves

$$\begin{array}{ccccc} H^1(M(c-2)) \otimes O_{\mathbf{P}^{r*}}(-1) & \to & H^1(M(c-1)) \otimes O_{\mathbf{P}^{r*}} & \to & 0 \\ \wr\| & & \wr\| & & \\ O^2_{\mathbf{P}^{r*}}(-1) & \longrightarrow & O_{\mathbf{P}^{r*}} & \longrightarrow & 0 \end{array}$$

which is impossible for $r \geq 2$. Thus for some hyperplane $H$, $c_{c-1,H} \neq 0$. However, by the result on codimensions, this implies $c_H \geq c$. Moreover, by the diagram (10),

$$c = c_H + \operatorname{codim}(W \cap I_d(H),\ I_d(H)).$$

We conclude that

$$\operatorname{codim}(W \cap I_d(H),\ I_d(H)) = 0,$$

so $W \supseteq I_d(H)$. This completes the inductive step and thus the proof of Theorem 2.

**Remark.** In [5], it was shown that for $W$ base-point free and of codimension $c$, the map $\mu_c$ is surjective. However, this result was used only in the case $c = d-3$, where it may be deduced from Gotzmann's result. Gotzmann's theorem is quite strong and ought to have other interesting applications. More generally, the standard monomial techniques of Macaulay, Gotzmann, Bayer, and Stillman seem likely to be widely useful in a variety of questions of this kind.

Returning to the proof of Theorem 1, let $\Sigma, S, W, F$ be as before. For $d \geq 5$, $c = d-3$, we know by Theorem 2 that $W \supseteq I_d(L)$ for some line $L$. If $L_1, L_2$ are two distinct lines in $\mathbf{P}^3$, then

$$I_d(L_1)\big|_{L_2} = H^0(\mathcal{O}_{L_2}(d)) \quad \text{if } L_1 \cap L_2 = \varphi,$$
$$I_d(L_1)\big|_{L_2} = H^0(\mathcal{O}_{L_2}(d) \otimes I_P) \quad \text{if } L_1 \cap L_2 = P,$$

and thus if $W \supseteq I_d(L_1) + I_d(L_2)$, then $c \leq 1$. So for each $S \in \Sigma$ there is a unique line $L_S$ such that $W \supseteq I_d(L_S)$. We thus have a natural map

$$\Sigma \xrightarrow{\pi} G(2,4), \qquad S \to L_S.$$

For each $L \in G(2,4)$, let $\Sigma_L = \pi^{-1}(L)$. If $\Sigma_L$ is nonempty, then $\operatorname{codim}(\Sigma_L, \Sigma) \leq 4$. Choose an $L$ with $\Sigma_L \neq \emptyset$. Let $W_L \subseteq W$ be the pullback of $T_S(\Sigma_L)$ to $H^0(O_{\mathbf{P}^3}(d))$. Choose $S$ to be a general point of any component of $\Sigma_L$, so that $\operatorname{codim}(W_L, W) \leq 4$ and $\operatorname{codim}(W_L \cap I_d(L), I_d(L)) \leq 4$ are locally constant on $\Sigma_L$ near $S$.

Since $W \supseteq J_d(F)$, the restriction of $J_d(F)$ to $L$ has codimension $\geq d-3$ in $H^0(\mathcal{O}_L(d))$. Since it is base-point free and

$$J_d(F)\big|_L = \operatorname{im}(J_{d-1}(F)\big|_L \otimes H^0(\mathcal{O}_L(1)) \to H^0(\mathcal{O}_L(d)))$$

we conclude from Gotzmann's theorem that

$$\operatorname{codim}(J_{d-1}(F)\big|_L, H^0(\mathcal{O}_L(d-1))) \geq d-2$$

and therefore

$$\dim(J_{d-1}(F)\big|_L) \leq 2.$$

Now, choose homogeneous coordinates $(z_0, \cdots, z_3)$ for $\mathbf{P}^3$ so that $L = \{z_0 = 0, z_1 = 0\}$. Let

$$\alpha = \dim \operatorname{span}(F_0\big|_L, F_1\big|_L)$$

where $F_i = \partial F/\partial z_i$. By choosing $S$ generally on any component of $\Sigma_L$, we may assume $\alpha$ is locally constant near $S$. We must deal separately with the cases $\alpha = 0, 1, 2$.

If $\alpha = 2$, then by a linear change of coordinates preserving the fact $L = \{z_0 = 0, z_1 = 0\}$, we may arrange that $F_2\big|_L = 0$ and $F_3\big|_L = 0$. Now

$$F\big|_L = (z_0F_0\big|_L + z_1F_1\big|_L + z_2F_2\big|_L + z_3F_3\big|_L)/d = 0$$

so $L \subseteq S$ and we are done, as now a general element of $\Sigma$ contains a line.

If $\alpha = 0$, then the equations $F_0\big|_L = 0$ and $F_1\big|_L = 0$ hold identically on the component of $\Sigma_L$ containing $S$. Thus for all $G \in W_L$,

$$G_0\big|_L = 0, \quad G_1\big|_L = 0.$$

Since $W \supseteq I_d(L)$, and $\operatorname{codim}(W_L, W) \leq 4$, we know that $G = z_0A + z_1B$ belongs to $W_L$ for a codimension$\leq 4$ subspace of

$$\{(A, B) \mid A, B \in H^0(O_{\mathbf{P}^r}(d-1))\}.$$

Now

$$G_0\big|_L = A\big|_L = 0, \quad G_1\big|_L = B\big|_L = 0 \quad \text{if } G \in W_L.$$

However,

$$\{(A, B) \mid A, B \in H^0(O_{\mathbf{P}^r}(d-1)), A\big|_L = 0, B\big|_L = 0\}$$

has codimension $2d$, so $\operatorname{codim}(W_L, W) \geq 2d$. This is a contradiction for $d \geq 3$.

The last case is $\alpha = 1$. We now have locally near $S$ on $\Sigma_L$ a family of equations

$$(a_0(t)F_0(t) + a_1(t)F_1(t))\big|_L = 0$$

as $t$ varies over $\Sigma_L$. Differentiating in the direction corresponding to $G \in W_L$ at $S$, we have

$$a_0(0)G_0\big|_L + a_1(0)G_1\big|_L = -a_0'(0)F_0\big|_L - a_1'(0)F_1\big|_L \in \operatorname{span}(F_0\big|_L, F_1\big|_L)$$

where $t = 0$ is the point of $\Sigma_L$ corresponding to $S$.

For $G = z_0A + z_1B$, we have

$$a_0(0)A\big|_L + a_1(0)B\big|_L \in \operatorname{span}(F_0\big|_L, F_1|L)$$

if $G \in W_L$. Since $\alpha = 1$,

$$\dim \operatorname{span}(F_0\big|_L, F_1\big|_L) = 1$$

and thus

$$\{(A, B) \mid A, B \in H^0(O_{\mathbf{P}^3}(d-1)), a_0(0)A\big|_L + a_1(0)B\big|_L \in \operatorname{span}(F_0\big|_L, F_1\big|_L)\}$$

has codimension $d - 1$. So

$$d - 1 \leq \operatorname{codim}(W_L \cap I_d(L), I_d(L)) \leq \operatorname{codim}(W_L, W) \leq 4$$

which is a contradiction if $d \geq 6$.

This reduces us to the case $d = 5$ and $\operatorname{codim}(W_L \cap I_d(L), I_d(L)) = 4$. Let $U = \operatorname{span}(G^1, G^2, G^3, G^4)$ be a 4-dimensional subspace of $W_L$ such that $G^1|_L, \cdots, G^4|_L$ are linearly independent. By a change of homogeneous coordinates on $\mathbf{P}^3$ keeping $L = \{z_0 = 0, z_1 = 0\}$, since $\alpha = 1$, we may arrange that $F_3|_L = 0$. This equation deforms to an equation

$$(a_0(t)F_0(t) + \cdots + a_3(t)F_3(t))\big|_L = 0$$

for $t \in \Sigma_L$ near $S$. If $t = 0$ corresponds to $S$, $(a_0(0), \cdots, a_3(0)) = (0,0,0,1)$. Differentiating in the direction corresponding to $G \in W_L$, we get

$$G_3\big|_L \in J_{d-1}(F)\big|_L.$$

In particular, since $\dim(J_{d-1}(F)\big|_L) \leq 2$, we may change the basis of $U$ so that $G_3^1|_L = 0$ and $G_3^2|_L = 0$. Since $z_2, z_3$ are homogeneous coordinates on $L$, we see that

$$G^1\big|_L, G^2\big|_L \in \operatorname{span}(z_2^d).$$

Thus some linear combination of $G^1$ and $G^2$ restricts to zero on $L$, which contradicts the assumption on $U$. This completes the proof of Theorem 1.

An interesting open problem concerns the case $d = 5$. Irreducible components of $\Sigma_5$ may have codimensions 2, 3, and 4. We have just shown that the only component having codimension 2 consists of quintics containing a line. One easily verifies that the quintics containing a plane conic gives a component of $\Sigma_5$ of codimension 3. Are there any others?* This relates to a problem suggested by Joe Harris: although $\Sigma_d$ has countably many components, there should be only finitely many whose codimension is smaller than the maximum value $\binom{d-1}{3}$.

I want to thank Joe Harris for some useful ideas, and for showing me his joint work with Ciro Ciliberto, which gives an intriguing alternative approach to proving Theorem 1 using a degeneration argument. I learned of the work of Macaulay and Gotzmann through the generous aid of Dave Bayer, David Eisenbud, and Tony Iarrobino.

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*__Added in proof.__ This has been solved by Claire Voisin in [8].

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